Modeling the transplant waiting list: A queueing model with reneging

Stefanos A. Zenios. Received 19 May 1998. Revised 14 September 1998.

# Introduction

In the United States, organs are allocated using a system that prioritizes transplant candidates according to a combination of waiting time and medical criteria. There is an ongoing controversy surrounding the observed differences in waiting times between different groups.

We propose a queueing model that provides a representation of the transplant waiting list. The model assumes that there are:

* several classes of patients who queue up for transplantation,
* several classes of organs which serve the patients,
* patients renege due to death while waiting for a transplant.

The model takes the form of a multiclass, multiserver queueing system with reneging, and the allocation policies take the form of scheduling rules for the servers. For simplicity the allocation policies considered here are open-loop randomized policies. (no feedback)

Our analysis focuses on two main objectives:

* identify the major causes for the observed differences in the waiting time between various demographic groups,
* identify policies that can eliminate such differences.

For the first, we pursue a queueing-based analysis of our model and develop closed-form expressions for three performance outcomes:

* average waiting time,
* average waiting time for patients who receive transplantation,
* fraction of patients who receive transplantation.

These expressions identify the main factors that can generate differences in the mean waiting times between various patient classes.

For the second, we analyze the closed-form expressions to identify simple randomized policies that eliminate between-class differences in the waiting times and analyze the impact of such policies on the other two outcomes.

# A multiclass queueing model with reneging

The model assumes that there are:

* *K* classes of transplant candidates who arrive according to independent Poisson processes with rate ;
* *J* classes of organs that arrive according to independent Poisson processes with rate ;

The class definitions are based on the demographic, immunological and physiological characteristics of the patients and donors.

Patients of each class renege from the system due to death after an exponentially distributed amount of time with rate *.* The allocation policy takes the form of a static randomized policy. In particular, is the fraction of class *j* organs that are allocated to patients of class *k*, and candidates of the same class are allocated organs on a first-come first-transplanted (FCFT) basis.

Define:

* the total patient arrival rate 🡪
* the total organ arrival rate 🡪
* the total organ allocation rate to patients of class *k* 🡪
* the mean patient death rate 🡪
* the traffic intensity (the organ demand-to-supply ratio) 🡪
* the demand-to-supply ratio for class *k* patients 🡪
* the ratio of the organ allocation rate for patients of class *k* over their respective death rate 🡪

We will also assume that:

* the demand for organs exceeds the supply ()
* the organ arrival rate is much larger than the individual patient death rates,.

Our multiclass queueing model is closely related to the classical multiclass, multiserver queueing system with reneging:

* the transplant patients correspond to customers,
* the organs correspond to servers,
* the departure of patients from the waiting list due to death corresponds to customer reneging,
* the allocation policy corresponds to the server scheduling rule.

The only difference is that in our model the patient-customers depart from the system when service begins and not when service ends: the service initiation epochs correspond to the arrival of an organ. Despite this difference, the queue-length process for our model is equivalent to the queue-length process for the traditional queueing model with reneging.

Our queueing model is also related to the queueing models with negative customers:

* positive customers (transplant candidates) behave as customers in traditional queueing models,
* negative customers (organs) play the role of a secondary service.

Our queueing model is equivalent to a multiclass queue with several classes of negative customers.

The assumption to model the allocation policies by simple randomization rules makes the queueing system decompose into *K* independent queueing systems, one for each patient class. Therefore, to do performance analysis for this queueing system, one can start by analyzing the simpler system that has one class of patients and one class of organs.

# Major performance results

We derive integral expressions for the following three performance outcomes for each patient class:

* expected stationary waiting time,
* expected stationary waiting time for patients who receive transplantation,
* stationary fraction of patients who receive transplantation.

The derivation of these integral expressions proceeds in three steps:

1. an expression is derived for the equilibrium distribution of the queue-length process.
2. using the equilibrium distribution we derive expressions for the Laplace transform of the stationary waiting time for each patient class, and the stationary waiting time for patients of each class who receive transplantation.
3. using simple properties of Laplace transforms, we derive integral expressions for the three performance outcomes.

Assuming that the allocation policy is *v* and that there exists a stationary distribution for the queue-length process, let:

* , the stationary number of class *k* patients in the system,
* , the stationary waiting time for patients of class *k*,
* , the stationary waiting time for patients of class *k* who receive transplantation,
* , the stationary fraction of class *k* patients who receive transplantation,
* , *n* ∈ N, the stationary distribution for the queue-length process,
* , the stationary *virtual waiting time* for patients of class *k*, that’s the stationary amount of time that elapses between the arrival of a class *k* patient and the instance that this patient would receive an organ (assuming that the patient will not have died by then); in the traditional queueing context, this variable gives the stationary *workload* from class *k* patients.

**Proposition 1.** The stationary probability distribution for the queue-length process under a randomized allocation policy *v* is:

🡪

and the expected stationary queue length is:

🡪

**Proposition 2.** Under a randomized allocation policy *v*, the Laplace transforms for the stationary virtual waiting time, the stationary waiting time and the stationary waiting time for patients who receive transplantation for each patient class *k* are as follows:

🡪

🡪

🡪

**Proposition 3.** Under the randomized policy *v*,

🡪

🡪

🡪

# Asymptotic expressions

We derive asymptotic expressions for:

* expected stationary waiting time,
* expected stationary waiting time for patients who receive transplantation,
* fraction of patients who receive transplantation for each patient class.

The asymptotic regime for these expressions assumes that:

* the patient and organ arrival rates become infinitely large,
* the reneging rates are maintained fixed,
* the patient arrival rate for each class exceeds the corresponding organ allocation rate.

Assuming that the allocation policy *v* is independent of the scaling factor *n*, define:

* the candidate arrival rates 🡪
* the organ arrival rates are 🡪
* the reneging rates are 🡪
* the organ allocation rates to each patient class 🡪

The asymptotic regime assumes that , and lets .

**Proposition 4.** The following asymptotics hold as :

🡪

🡪

🡪

The first expression implies that differences in the expected stationary waiting times between various classes can be caused either by differences in the allocation rates or differences in the mortality rates between these classes. The latter provides an additional factor that may contribute to the longer waiting time for African-Americans. Specifically, the mortality rates for African-Americans on the waiting list are lower than Caucasians. Therefore, African-Americans would have longer mean waiting times than Caucasians, even if they had the same allocation rates.

The following proposition identifies the policy that can equalize the expected stationary waiting times for all patient classes and examines the impact of that policy on the mean waiting time of patients of each class who receive transplantation.

**Proposition 5.** As , the expected stationary waiting times for all patient classes can be asymptotically equalized if and only if . The static policy that equalizes the expected stationary waiting times for all classes is given by:

🡪

Furthermore, under such a policy, the expected stationary waiting time for patients of class *k* who receive transplantation is:

🡪

To equalize the mean stationary waiting times for all patient classes, one must allocate a higher percentage of organs to the patients with the lowest mortality rate. However, this is done at the expense of differences in the mean waiting times for patients of each class who receive transplantation and the fraction of patients of each class who receive transplantation.

One should not evaluate an allocation policy based only on its impact on waiting times, but consider at least two of the three outcomes considered in this report. In particular, consider the mean waiting times for each patient class, which include the confounding effect of mortality, and the fraction of patients of each class who receive transplantation, which isolate the confounding effect of mortality when organs are allocated using a randomized rule.

# Concluding remarks

The model captures some of the major effects that underlie the organ allocation problem. This simple queueing model hold for a more general queueing model with non-exponential reneging and time inhomogeneous Poisson arrivals.

The identification of the mortality rate as one of the main driving forces behind the performance of the allocation policies motivates a new hypothesis that can explain African-Americans to have longer waiting times than Caucasians. Results show that in order to evaluate the performance of an allocation policy, one should consider both the impact of the policy on waiting times and on the percentage of candidates who receive transplantation. Focusing on waiting time alone can be misleading because the patient mortality serves as a confounding factor.